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Strong Convergence of Averaged Approximants for Asymptotically Nonexpansive Mappings in Banach Spaces

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Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. In this paper, we show that $F(T)$ is a sunny, nonexpansive retract of C. Using this result, we discuss the strong convergence of the sequence $\{x_n\}$ defined by $x_n = a_n x + (1 - a_n)\, 1/(n+1) \sum_{j=0}^{n} T^j x_n$ for $n = 0, 1, 2, \ldots$, where $x \in C$ and $\{a_n\}$ is a real sequence in $(0, 1]$. © 1999 Academic Press

1. INTRODUCTION

Let C be a subset of a Banach space. A mapping T from C into E is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for each $x, y \in C$. A mapping T from C into itself is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ such that $\overline{\lim}_n k_n \leqslant 1$ and $\|T^n x - T^n y\| \leqslant k_n \|x - y\|$ for each $x, y \in C$ and $n = 0, 1, 2, \ldots$.

Let C be a closed, convex subset of a Banach space E. Let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty, let x be an element of C and for each t with $0 < t < 1$, let x_t be the unique point of C which satisfies $x_t = tx + (1-t)\, Tx_t$. Browder

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[2] showed that $\{x_t\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ as $t \downarrow 0$ in the case when E is a Hilbert space. Reich [8] extended Browder's result to the case when E is a uniformly smooth Banach space and he showed that $F(T)$ is a sunny, nonexpansive retract of C, i.e., there exists a nonexpansive retraction P from C onto $F(T)$ such that $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geqslant 0$ with $Px + t(x - Px) \in C$. Recently, using an idea of Browder [2], Shimizu and Takahashi [10] studied the convergence of another approximating sequence for an asymptotically nonexpansive mapping. Let T be an asymptotically nonexpansive mapping with Lipschitz constants $\{k_n\}$ such that the set $F(T)$ of fixed points of T is nonempty. Let $0 < a < 1$, let $b_n = 1/n \sum_{j=1}^{n} (1 + |1 - k_j| + e^{-j})$ and let $a_n = (b_n - 1)/(b_n - 1 + a)$ for $n = 1, 2, \ldots$. Let x be an element of C and let x_n be the unique point of C which satisfies $x_n = a_n x + (1 - a_n)\, 1/n \sum_{j=1}^{n} T^j x_n$ for $n = 1, 2, \ldots$. They showed that $\{x_n\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ in the case when E is a Hilbert space.

In this paper, we extend Shimizu and Takahashi's result to a Banach space. For an asymptotically nonexpansive mapping T, we show that the set $F(T)$ of fixed points of T is a sunny, nonexpansive retract of C and the sequence $\{x_n\}$ defined above converges strongly to an element of $F(T)$. Our results are the following:

THEOREM 1. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. Then $F(T)$ is a sunny, nonexpansive retract of C.*

THEOREM 2. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n\}$ such that the set $F(T)$ of fixed points of T is nonempty and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence such that*

$$0 < a_n \leqslant 1, \qquad \lim_{n \to \infty} a_n = 0, \qquad \text{and} \qquad \overline{\lim_{n \to \infty}} \frac{b_n - 1}{a_n} < 1,$$

where $b_n = 1/(n+1) \sum_{j=0}^{n} k_j$ for $n = 0, 1, \ldots$. Let x be an element of C and let x_n be the unique point of C which satisfies

$$x_n = a_n x + (1 - a_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \tag{1.1}$$

for $n \geqslant N_0$, *where* N_0 *is a sufficiently large natural number. Then* $\{x_n\}$ *converges strongly to* Px.

Remark. The inequality $\overline{\lim}_n (b_n - 1)/a_n < 1$ implies that there exists a natural number N_0 such that $(1 - a_n)\, b_n < 1$ for $n \geqslant N_0$. So for $n \geqslant N_0$, there exists the unique point x_n of C which satisfies (1.1), since the mapping T_n from C into itself defined by $T_n u = a_n x + (1 - a_n)\, 1/(n+1) \sum_{j=0}^{n} T^j u$ satisfies $\|T_n u - T_n v\| \leqslant (1 - a_n)\, b_n \,\|u - v\|$ for each $u, v \in C$.

In the case when T is nonexpansive, we have the following:

THEOREM 3. *Let* C *be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let* T *be a nonexpansive mapping from* C *into itself such that the set* $F(T)$ *of fixed points of* T *is nonempty and let* P *be the sunny, nonexpansive retraction from* C *onto* $F(T)$. *Let* $\{a_n\}$ *be a real sequence such that* $0 < a_n \leqslant 1$ *and* $a_n \to 0$. *Let* x *be an element of* C *and let* x_n *be the unique point of* C *which satisfies* (1.1) *for* $n = 0, 1, \ldots$. *Then* $\{x_n\}$ *converges strongly to* Px.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$, the set of all nonnegative integers. For a real number a, we also denote $\max\{a, 0\}$ by $(a)_+$. We denote by Δ^n, the set $\{\lambda = (\lambda_0, \ldots, \lambda_n): \lambda_i \geqslant 0, \sum_{j=0}^{n} \lambda_j = 1\}$ for $n \in \mathbb{N}$. For a subset C of a Banach space, we denote by $\operatorname{co} C$, the convex hull of C.

Let E be a Banach space and let $r > 0$. We denote by B_r, the closed ball in E with center 0 and radius r. E is said to be uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x + y)/2\| \leqslant 1 - \delta$ for each $x, y \in B_1$ with $\|x - y\| \geqslant \varepsilon$. Let C be a subset of E, let T be a mapping from C into E and let $\varepsilon > 0$. By $F(T)$ and $F_\varepsilon(T)$, we mean the sets $\{x \in C: x = Tx\}$ and $\{x \in C: \|x - Tx\| \leqslant \varepsilon\}$, respectively. Let $k \geqslant 0$. We denote by $\operatorname{Lip}(C, k)$, the set of all mappings from C into E satisfying $\|Tx - Ty\| \leqslant k\, \|x - y\|$ for each $x, y \in C$. We remark that $\operatorname{Lip}(C, 1)$ is the set of all nonexpansive mappings from C into E. The following is a useful proposition due to Bruck [5]:

PROPOSITION 1. *Let* C *be a closed, convex subset of a uniformly convex Banach space. Then for each* $R > 0$, *there exists a strictly increasing, convex, continuous function* $\gamma\colon [0, \infty) \to [0, \infty)$ *such that* $\gamma(0) = 0$ *and*

$$\gamma\left(\left\| T\left(\sum_{j=0}^{n} \lambda_j x_j\right) - \sum_{j=0}^{n} \lambda_j Tx_j \right\|\right) \leqslant \max_{0 \leqslant j < k \leqslant n} (\|x_j - x_k\| - \|Tx_j - Tx_k\|)$$

for all $n \in \mathbb{N}$, $\lambda \in \Delta^n$, $x_0, \ldots, x_n \in C \cap B_R$, *and* $T \in \operatorname{Lip}(C, 1)$.

Let μ be a continuous, linear functional on l^∞ and let $(a_0, a_1, ...) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, ...))$. We call μ a Banach limit [1] when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, ...) \in l^\infty$. For a Banach limit, we know that

$$\underline{\lim}_{n \to \infty} a_n \leqslant \mu_n(a_n) \leqslant \overline{\lim}_{n \to \infty} a_n \qquad \text{for all} \quad (a_0, a_1, ...) \in l^\infty. \tag{2.1}$$

We also know the following from Lemma in [11] and its proof; see also [9, pp. 314–315]:

Proposition 2. *Let C be a closed, convex subset of a uniformly convex Banach space E. Let $\{x_n\}$ be a bounded sequence of E, let μ be a Banach limit and let g be a real valued function on C defined by*

$$g(y) = \mu_n \|x_n - y\|^2 \qquad \textit{for each} \quad y \in C.$$

Then g is continuous and convex, and g satisfies $\lim_{\|y\| \to \infty} g(y) = \infty$. Moreover, for each $R > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$g\left(\frac{y+z}{2}\right) \leqslant \frac{g(y) + g(z)}{2} - \delta$$

for all $y, z \in C \cap B_R$ with $\|y - z\| \geqslant \varepsilon$.

Let E' be the topological dual of E. The value of $y \in E'$ at $x \in E$ will be denoted by $\langle x, y \rangle$. We also denote by J, the duality mapping from E into $2^{E'}$, i.e.,

$$Jx = \{ y \in E' : \langle x, y \rangle = \|x\|^2 = \|y\|^2 \} \qquad \text{for each} \quad x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.2) exists uniformly for $x \in U$. E is said to be uniformly smooth if the limit (2.2) exists uniformly for $x, y \in U$. It is well known that if E is smooth then the duality mapping is single-valued and norm to weak star continuous. In the case when the norm of E is uniformly Gâteaux differentiable, we know the following from [12, Lemma 1]; see also [6, p. 586]:

PROPOSITION 3. *Let C be a convex subset of a Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded subset of E, let z be a point of C and let μ be a Banach limit. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n \langle y - z, J(x_n - z) \rangle \leqslant 0 \qquad \text{for all} \quad y \in C.$$

Let C be a convex subset of E, let K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geqslant 0$ with $Px + t(x - Px) \in C$. If the sunny retraction P is also nonexpansive, then K is said to be a sunny, nonexpansive retract of C. Concerning sunny, nonexpansive retractions, we know the following [3, 7]:

PROPOSITION 4. *Let C be a convex subset of a smooth Banach space, let K be a nonempty subset of C and let P be a retraction from C onto K. Then P is sunny and nonexpansive if and only if*

$$\langle x - Px, J(y - Px) \rangle \leqslant 0 \qquad \text{for all} \quad x \in C \quad \text{and} \quad y \in K.$$

Hence there is at most one sunny, nonexpansive retraction from C onto K.

3. PROOF OF THEOREMS

To prove Lemmas 1, 2, 3 below, we use the methods employed in [4, 5].

LEMMA 1. *Let C be a closed, convex subset of a uniformly convex Banach space. Then for each $R > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$(\mathrm{co}(F_\eta(T) \cap B_R) + B_\eta) \cap C \subset F_\varepsilon(T)$$

for all $T \in \mathrm{Lip}(C, 1 + \eta)$.

Proof. Let $R > 0$. Then there exists a function γ which satisfies the conditions in Proposition 1. Let $\varepsilon > 0$. Choose $\eta > 0$ such that $(3 + \eta)\,\eta + (1 + \eta)\,\gamma^{-1}(2(1 + R)\,\eta) \leqslant \varepsilon$. Let $T \in \mathrm{Lip}(C, 1 + \eta)$. Pick $\lambda \in \Delta^n$, $x_0, ..., x_n \in F_\eta(T) \cap B_R$ and $y \in B_\eta$ such that $\sum_{i=0}^{n} \lambda_i x_i + y \in C$. Since $1/(1 + \eta)\,T \in \mathrm{Lip}(C, 1)$, we have

$$\begin{aligned}
&\gamma\left(\frac{1}{1+\eta}\left\|T\left(\sum_{i=0}^{n}\lambda_i x_i\right)-\sum_{i=0}^{n}\lambda_i T x_i\right\|\right)\\
&\quad\leqslant \max_{0\leqslant i<j\leqslant n}\left(\|x_i-x_j\|-\frac{1}{1+\eta}\|Tx_i-Tx_j\|\right)\\
&\quad\leqslant \max_{0\leqslant i<j\leqslant n}\left(\|x_i-Tx_i\|+\|x_j-Tx_j\|+\frac{\eta}{1+\eta}\|Tx_i-Tx_j\|\right)\\
&\quad\leqslant 2(1+R)\,\eta.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\left\|\left(\sum_{i=0}^{n}\lambda_i x_i+y\right)-T\left(\sum_{i=0}^{n}\lambda_i x_i+y\right)\right\|\\
&\quad\leqslant \|y\|+\left\|\sum_{i=0}^{n}\lambda_i x_i-\sum_{i=0}^{n}\lambda_i T x_i\right\|\\
&\qquad+\left\|\sum_{i=0}^{n}\lambda_i T x_i-T\left(\sum_{i=0}^{n}\lambda_i x_i\right)\right\|+\left\|T\left(\sum_{i=0}^{n}\lambda_i x_i\right)-T\left(\sum_{i=0}^{n}\lambda_i x_i+y\right)\right\|\\
&\quad\leqslant (3+\eta)\,\eta+(1+\eta)\,\gamma^{-1}(2(1+R)\,\eta)\leqslant\varepsilon. \quad\blacksquare
\end{aligned}$$

LEMMA 2. *Let C be a closed, convex subset of a uniformly convex Banach space. Then for each $p\in\mathbb{N}$, $R>0$ and $\varepsilon>0$, there exist $\eta>0$ and $N\in\mathbb{N}$ such that for each pair $T\in \mathrm{Lip}(C, 1+\eta)$ and $\{x_{j,n}\colon n\in\mathbb{N},\ j=0, ..., p\}\subset C\cap B_R$ satisfying*

$$\frac{1}{n+1}\sum_{i=0}^{n}\|x_{j,i+1}-Tx_{j,i}\|\leqslant\eta \qquad \text{for all}\quad n\geqslant N \quad\text{and}\quad j=0, ..., p, \tag{3.1}$$

there holds

$$\frac{1}{n+1}\sum_{i=0}^{n}\left\|\sum_{j=0}^{p}\lambda_j x_{j,i+1}-T\left(\sum_{j=0}^{p}\lambda_j x_{j,i}\right)\right\|\leqslant\varepsilon \qquad \text{for all}\quad n\geqslant N \quad\text{and}\quad \lambda\in\Delta^p.$$

Proof. Let $R>0$. Then there exists a function γ which satisfies the conditions in Proposition 1. Let $p\in\mathbb{N}$ and let $\varepsilon>0$. Then there exist $\eta>0$ and $N\in\mathbb{N}$ satisfying

$$\eta+(1+\eta)\,\gamma^{-1}\left(\frac{p(p+1)}{2}\left(\frac{2R}{N+1}+2(1+R)\,\eta\right)\right)\leqslant\varepsilon.$$

Pick $T \in \operatorname{Lip}(C, 1+\eta)$ and $\{x_{j,i} : i \in \mathbb{N},\ j = 0, ..., p\} \subset C \cap B_R$ satisfying (3.1). Let $n \geqslant N$ and $\lambda \in \Delta^p$. Since

$$-\frac{1}{1+\eta} \|Tx_{j,i} - Tx_{k,i}\| \leqslant -\|x_{j,i+1} - x_{k,i+1}\| + \|x_{j,i+1} - Tx_{j,i}\| + \frac{\eta}{1+\eta} \|Tx_{j,i} - Tx_{k,i}\| + \|Tx_{k,i} - x_{k,i+1}\|,$$

we get

$$\begin{aligned}
&\gamma\left(\frac{1}{n+1} \sum_{i=0}^{n} \frac{1}{1+\eta} \left\| \sum_{j=0}^{p} \lambda_j Tx_{j,i} - T\left(\sum_{j=0}^{p} \lambda_j x_{j,i}\right)\right\|\right) \\
&\leqslant \frac{1}{n+1} \sum_{i=0}^{n} \gamma\left(\frac{1}{1+\eta} \left\| \sum_{j=0}^{p} \lambda_j Tx_{j,i} - T\left(\sum_{j=0}^{p} \lambda_j x_{j,i}\right)\right\|\right) \\
&\leqslant \frac{1}{n+1} \sum_{i=0}^{n} \max_{0 \leqslant j < k \leqslant p} \left(\|x_{j,i} - x_{k,i}\| - \frac{1}{1+\eta} \|Tx_{j,i} - Tx_{k,i}\|\right) \\
&\leqslant \frac{1}{n+1} \sum_{i=0}^{n} \sum_{0 \leqslant j < k \leqslant p} \left(\|x_{j,i} - x_{k,i}\| - \frac{1}{1+\eta} \|Tx_{j,i} - Tx_{k,i}\|\right) \\
&\leqslant \sum_{0 \leqslant j < k \leqslant p} \left(\frac{\|x_{j,0} - x_{k,0}\| - \|x_{j,n+1} - x_{k,n+1}\|}{n+1} + 2(1+R)\eta\right) \\
&\leqslant \frac{p(p+1)}{2} \left(\frac{2R}{N+1} + 2(1+R)\eta\right).
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\frac{1}{n+1} \sum_{i=0}^{n} \left\| \sum_{j=0}^{p} \lambda_j x_{j,i+1} - T\left(\sum_{j=0}^{p} \lambda_j x_{j,i}\right)\right\| \\
&\leqslant \sum_{j=0}^{p} \lambda_j \left(\frac{1}{n+1} \sum_{i=0}^{n} \|x_{j,i+1} - Tx_{j,i}\|\right) \\
&\quad + \frac{1}{n+1} \sum_{i=0}^{n} \left\| \sum_{j=0}^{p} \lambda_j Tx_{j,i} - T\left(\sum_{j=0}^{p} \lambda_j x_{j,i}\right)\right\| \\
&\leqslant \eta + (1+\eta)\, \gamma^{-1}\left(\frac{p(p+1)}{2} \left(\frac{2R}{N+1} + 2(1+R)\eta\right)\right) \leqslant \varepsilon. \quad \blacksquare
\end{aligned}$$

The following is crucial to the proof of our theorems:

THEOREM 3. *Let C be a closed, convex subset of a uniformly convex Banach space. Then for each $r>0$, $R\geqslant r$ and $\varepsilon>0$, there exist $\eta>0$ and $N\in\mathbb{N}$ such that for each $l\in\mathbb{N}$ and for each mapping T from C into itself satisfying $\sup\{\|T^n x\|: n\in\mathbb{N}, x\in C\cap B_r\}\leqslant R$ and $T^l\in \operatorname{Lip}(C, 1+\eta)$, there holds*

$$\left\|\frac{1}{m+1}\sum_{i=0}^{m} T^i x - T^l\left(\frac{1}{m+1}\sum_{i=0}^{m} T^i x\right)\right\|\leqslant\varepsilon$$

for all $m\geqslant lN$ and $x\in C\cap B_r$. Especially, for each $r>0$ and for each asymptotically nonexpansive mapping T from C into itself with $F(T)\neq\emptyset$,

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}}\ \sup_{x\in C\cap B_r}\left\|\frac{1}{m+1}\sum_{i=0}^{m} T^i x - T^l\left(\frac{1}{m+1}\sum_{i=0}^{m} T^i x\right)\right\| = 0.$$

Proof. Let $r>0$, let $R\geqslant r$ and let $\varepsilon>0$. By Lemma 1, there exist $\delta>0$ and $\xi>0$ such that

$$(\operatorname{co}(F_\delta(S)\cap B_R)+B_\delta)\cap C\subset F_\varepsilon(S)\qquad\text{for all}\quad S\in\operatorname{Lip}(C, 1+\delta)$$

and

$$(\operatorname{co}(F_\xi(S)\cap B_R)+B_\xi)\cap C\subset F_\delta(S)\qquad\text{for all}\quad S\in\operatorname{Lip}(C, 1+\xi).$$

Choose $\tau>0$ and $p\in\mathbb{N}$ such that $R\tau\leqslant\xi/3$, $\tau\leqslant\xi$ and $2R/(p+1)\leqslant\tau^2/2$. By Lemma 2, there exist $\eta>0$ and $N\in\mathbb{N}$ such that for each $S\in\operatorname{Lip}(C, 1+\eta)$ and $\{x_{j,n}: n\in\mathbb{N}, j=0, ..., p\}\subset C\cap B_R$ satisfying

$$\frac{1}{n+1}\sum_{i=0}^{n}\|x_{j,i+1}-Sx_{j,i}\|\leqslant\eta\qquad\text{for all}\quad n\geqslant N\quad\text{and}\quad j=0, ..., p,$$

there holds

$$\frac{1}{n+1}\sum_{i=0}^{n}\left\|\sum_{j=0}^{p}\lambda_j x_{j,i+1}-S\left(\sum_{j=0}^{p}\lambda_j x_{j,i}\right)\right\|\leqslant\frac{\tau^2}{2}\qquad\text{for all}\quad n\geqslant N\quad\text{and}\quad\lambda\in\Delta^p.$$

We may assume $\eta\leqslant\xi$ and $PR/(N+1)\leqslant\xi/3$. Let $l\in\mathbb{N}$ and let T be a mapping from C into itself satisfying $\sup\{\|T^n x\|: n\in\mathbb{N}, x\in C\cap B_r\}\leqslant R$ and $T^l\in\operatorname{Lip}(C, 1+\eta)$. We may assume $l\neq 0$. Let $x\in C\cap B_r$. Set $y_n^q=T^{q+nl}x$ for $n\in\mathbb{N}$ and $q=0, ..., l-1$. We remark from the hypothesis of T that $\|y_n^q\|\leqslant R$ for $n\in\mathbb{N}$ and $q=0, ..., l-1$. Put $w_i^q=1/(p+1)\sum_{j=0}^{p} y_{j+i}^q$ for $i\in\mathbb{N}$ and $q=0, 1, ..., l-1$. Let $n\geqslant N$ and let $q\in\{0, 1, ..., l-1\}$. Since $y_{j+i+1}^q=T^l y_{j+i}^q$ for $j=0, 1, ..., p$, we get

$$\frac{1}{n+1}\sum_{i=0}^{n}\|w_i^q - T^l w_i^q\| \leqslant \frac{1}{n+1}\sum_{i=0}^{n}\|w_i^q - w_{i+1}^q\| + \frac{1}{n+1}\sum_{i=0}^{n}\|w_{i+1}^q - T^l w_i^q\|$$
$$\leqslant \frac{2R}{p+1} + \frac{\tau^2}{2} \leqslant \tau^2.$$

Set $A_n^q = \{i \in \{0, ..., n\} : \|w_i^q - T^l w_i^q\| \geqslant \tau\}$ and $B_n^q = \{0, ..., n\} \backslash A_n^q$. Then we have $\# A_n^q/(n+1) \leqslant \tau$, where $\# A_n^q$ is the cardinality of the set A_n^q. Since

$$\left\|\frac{1}{n+1}\sum_{i=0}^{n} y_i^q - \frac{1}{n+1}\sum_{i=0}^{n} w_i^q\right\|$$
$$\leqslant \frac{1}{p+1}\sum_{j=0}^{p}\left\|\frac{1}{n+1}\sum_{i=0}^{n} T^{q+il}x - \frac{1}{n+1}\sum_{i=0}^{n} T^{q+(j+i)l}x\right\|$$
$$\leqslant \frac{1}{p+1}\sum_{j=0}^{p}\frac{2jR}{n+1} = \frac{pR}{n+1},$$

we have

$$\left\|\frac{1}{n+1}\sum_{i=0}^{n} y_i^q - \frac{1}{\# B_n^q}\sum_{i\in B_n^q} w_i^q\right\|$$
$$\leqslant \left\|\frac{1}{n+1}\sum_{i=0}^{n} y_i^q - \frac{1}{n+1}\sum_{i=0}^{n} w_i^q\right\| + \left\|\frac{1}{n+1}\sum_{i\in A_n^q} w_i^q\right\|$$
$$+ \left\|\frac{1}{n+1}\sum_{i\in B_n^q} w_i^q - \frac{1}{\# B_n^q}\sum_{i\in B_n^q} w_i^q\right\|$$
$$\leqslant \frac{p}{n+1}R + \frac{\# A_n^q}{n+1}R + \frac{\# A_n^q}{n+1}R \leqslant \xi.$$

So by $1/\# B_n^q \sum_{i\in B_n^q} w_i^q \in \mathrm{co}\, F_\xi(T^l) \cap B_R$, we get

$$\frac{1}{n+1}\sum_{i=0}^{n} y_i^q \in (\mathrm{co}(F_\xi(T^l) \cap B_R) + B_\xi) \cap C \subset F_\delta(T^l)$$

for all $n \geqslant N$ and $q = 0, 1, ..., l-1$. Let $m \geqslant l(N+1)$. Choose $n \in \mathbb{N}$ and $s \in \{0, ..., l-2\}$ such that $m = l(n+1) + s$. Then $n \geqslant N$. Hence we obtain

$$\frac{1}{m+1}\sum_{i=0}^{m} T^i x = \frac{n+2}{m+1}\sum_{q=0}^{s}\left(\frac{1}{n+2}\sum_{i=0}^{n+1} y_i^q\right) + \frac{n+1}{m+1}\sum_{q=s+1}^{l-1}\left(\frac{1}{n+1}\sum_{i=0}^{n} y_i^q\right)$$
$$\in \mathrm{co}(F_\delta(T^l) \cap B_R) \cap C \subset F_\varepsilon(T^l)$$

for all $m \geqslant l(N+1)$ and $x \in C \cap B_r$. ∎

In the rest of this section, we assume that C, T, $\{k_n\}$, $\{a_n\}$, $\{b_n\}$, x and $\{x_n\}$ are as in Theorem 2, we set $a = \overline{\lim}_n (b_n - 1)/a_n$ and we set $x_n = x$ for $n = 0, 1, ..., N_0 - 1$.

LEMMA 4. *Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and let μ be a Banach limit. Then there exists the unique element z of C satisfying*

$$\mu_i \|x_{n_i} - z\|^2 = \min_{y \in C} \mu_i \|x_{n_i} - y\|^2 \tag{3.2}$$

and the point z is a fixed point of T.

Proof. From Proposition 2, it is easy to see that there exists the unique element z of C satisfying (3.2). If we can show $\lim_{n \to \infty} T^n z = z$, then z is a fixed point of T. Suppose $\lim_n T^n z \neq z$. Then there exists $\varepsilon > 0$ such that for each $m \in \mathbb{N}$, there exists $l \geqslant m$ satisfying $\|T^l z - z\| \geqslant \varepsilon$. Set $R = \sup\{\|T^m z\| : m \in \mathbb{N}\}$. By Proposition 2, there exists $\delta > 0$ such that

$$\mu_i \left\| x_{n_i} - \frac{x+y}{2} \right\|^2 \leqslant \frac{1}{2} (\mu_i \|x_{n_i} - x\|^2 + \mu_i \|x_{n_i} - y\|^2) - \delta \tag{3.3}$$

for all $x, y \in C \cap B_R$ with $\|x - y\| \geqslant \varepsilon$. By the property of ε, $\overline{\lim}_l k_l \leqslant 1$ and Lemma 3, there also exists $l \in \mathbb{N}$ such that $\|T^l z - z\| \geqslant \varepsilon$, $(k_l^2 - 1) \mu_i \|x_{n_i} - z\|^2 < \delta$ and $\mu_i \|x_{n_i} - T^l z\|^2 \leqslant \mu_i \|T^l x_{n_i} - T^l z\|^2 + \delta$. From (3.3), we have

$$\begin{aligned} \mu_i \left\| x_{n_i} - \frac{T^l z + z}{2} \right\|^2 &\leqslant \frac{1}{2} (\mu_i \|x_{n_i} - T^l z\|^2 + \mu_i \|x_{n_i} - z\|^2) - \delta \\ &\leqslant \mu_i \|x_{n_i} - z\|^2 + \frac{1}{2} ((k_l^2 - 1) \mu_i \|x_{n_i} - z\|^2 - \delta) \\ &< \mu_i \|x_{n_i} - z\|^2. \end{aligned}$$

So we get a contradiction. This completes the proof. ∎

LEMMA 5.

$$\langle x_n - x, J(x_n - z) \rangle \leqslant \frac{(b_n - 1)_+}{a_n} \|x_n - z\|^2 \qquad \textit{for all} \quad n \geqslant N_0 \quad \textit{and} \quad z \in F(T).$$

Proof. Let $n \geqslant N_0$ and let $z \in F(T)$. Since $a_n(x_n - x) = (1 - a_n)(1/(n+1) \sum_{j=0}^n T^j x_n - x_n)$ and $z \in F(T)$, we get

$$\begin{aligned}\langle x_n - x, J(x_n - z)\rangle &= \frac{1-a_n}{a_n}\left\langle \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n - x_n, J(x_n - z)\right\rangle \\ &= \frac{1-a_n}{a_n}\left(\left\langle \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n - \frac{1}{n+1}\sum_{j=0}^{n} T^j z, J(x_n - z)\right\rangle \right. \\ &\quad \left. + \langle z - x_n, J(x_n - z)\rangle \right) \\ &\leqslant \frac{1-a_n}{a_n}\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j \|x_n - z\|^2 - \|x_n - z\|^2\right) \\ &\leqslant \frac{(b_n - 1)_+}{a_n}\|x_n - z\|^2. \quad \blacksquare\end{aligned}$$

LEMMA 6. *Each subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *contains a subsequence of* $\{x_{n_i}\}$ *converging strongly to an element of* $F(T)$.

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and let μ be a Banach limit. There exists $z \in F(T)$ satisfying (3.2). By Lemma 5, we get $\mu_i \langle x_{n_i} - x, J(x_{n_i} - z)\rangle \leqslant (a)_+ \mu_i \|x_{n_i} - z\|^2$. This inequality and Proposition 3 yield

$$\mu_i \|x_{n_i} - z\|^2 \leqslant \frac{1}{1-(a)_+}\mu_i \langle x - z, J(x_{n_i} - z)\rangle \leqslant 0.$$

By (2.1), there exists a subsequence of $\{x_{n_i}\}$ converging strongly to z. $\blacksquare$

Now we can prove our theorems.

Proof of Theorem 1. Taking, for example,

$$a_n = \begin{cases} \dfrac{1}{n+1} & \text{if } \; b_n \leqslant 1, \\ \sqrt{b_n - 1} & \text{if } \; 1 < b_n \leqslant 2, \\ 1 & \text{if } \; 2 < b_n, \end{cases}$$

we may assume $a \leqslant 0$ only in this proof. First we shall show that $\{x_n\}$ converges strongly to an element of $F(T)$. By Lemma 6, we know that each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ contains a subsequence of $\{x_{n_i}\}$ converging strongly to an element of $F(T)$. Let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ converging strongly to elements y and z of $F(T)$, respectively. We shall show $y = z$. From Lemma 5, we have $\langle x_{n_i} - x, J(x_{n_i} - z)\rangle \leqslant (b_{n_i} - 1)_+ / a_{n_i} \|x_{n_i} - z\|^2$. So we get $\langle y - x, J(y - z)\rangle \leqslant 0$. By the same argument, we have $\langle z - x, J(z - y)\rangle \leqslant 0$. Adding these inequalities, we get $\|y - z\|^2 \leqslant 0$, i.e., $y = z$. So $\{x_n\}$ converges strongly to an element of $F(T)$. Hence we can define a mapping P from C onto $F(T)$ by $Px = \lim_{n\to\infty} x_n$, since x is an arbitrary point of C. By the argument above, we have $\langle x - Px, J(z - Px)\rangle \leqslant 0$ for all $x \in C$ and $z \in F(T)$. Therefore P is the sunny, nonexpansive retraction by Proposition 4. ∎

Proof of Theorem 2. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging strongly to an element y of $F(T)$. We shall show $y = Px$. By Lemma 5, we have $\langle x_{n_i} - x, J(x_{n_i} - Px)\rangle \leqslant (b_{n_i} - 1)_+ / a_{n_i} \, \|x_{n_i} - Px\|^2$. So we get $\langle y - x, J(y - Px)\rangle \leqslant (a)_+ \, \|y - Px\|^2$. Hence we obtain

$$(1 - (a)_+) \, \|y - Px\|^2 \leqslant \langle x - Px, J(y - Px)\rangle \leqslant 0$$

by Proposition 4. From $a < 1$, we have $y = Px$. Hence by Lemma 6, $\{x_n\}$ converges strongly to Px. ∎

Proof of Theorem 3. Since T is nonexpansive, we have $k_n = 1$ for all $n \in \mathbb{N}$ and hence $\overline{\lim}_n \, (b_n - 1)/a_n = 0 < 1$. So we obtain the desired result by Theorem 2. ∎

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